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Locally residual currents and Dolbeault cohomology on projective manifolds

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Abstract

Let X be a projective manifold of dimension n, and n hypersurfaces Y_i $(1 \leqslant i \leqslant n)$ on X, defining ample line bundles, in complete intersection position. After introducing sheaves of locally residual currents, we enunciate the following two main theorems. First, for any positive integer i, the Dolbeault cohomology group $H^i(\Omega_X^q)$ of the sheaf of holomorphic q-forms on X can be computed as the ith cohomology group of some complex of residual currents on X. We get from this the theorem of [A. Dickenstein, M. Herrera, C. Sessa, On the global liftings of meromorphic forms, Manuscripta Math. 47 (1984) 31–54] that any locally residual current on X which is $\bar{\partial}$-exact is globally residual. Secondly, let us assume that $Y_1 \cap \cdots \cap Y_p$ $(1 \leqslant p \leqslant n)$ are reduced complete intersections. We get another exact sequence computing $H^i(\Omega_X^n)$ by restricting to residual currents obtained from meromorphic forms with simple poles on the Y_i. We deduce from this a reformulation of the main theorem of [B. Khesin, A. Rosly, R. Thomas, A polar De Rham theorem, Topology 43 (2004) 1231–1246], saying that we can compute the cohomology groups $H^i(\Omega_X^n)$ by the cohomology of a complex of principal value currents. We also deduce from this the result from [P. Griffiths, Variations on a theorem of Abel, Invent. Math. 35 (1976) 321–390] that if $Y_1 \cap \cdots \cap Y_n$ is a set of distinct points $\{P_1, \ldots, P_s\}$, then for any sequence of s complex numbers c_i $(1 \leqslant i \leqslant s)$, there is a global meromorphic n-form Ψ with simple poles on each Y_i such that:

$$(\forall i, 1 \leqslant i \leqslant s) \quad \operatorname{Res}_{Y_1,\ldots,Y_n}^{P_i} \Psi = c_i$$

iff $\sum_{i=1}^{s} c_i = 0$. We give proofs of the theorems by mean of several exact sequences of sheaves of locally residual currents. We conclude by giving an application to the Hodge conjecture, giving some new formulation using our theorems.

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1. Main results

Let us introduce the main objects used in the results, that is locally residual currents, by giving a brief account of their construction. Further details can be found in [2], or [1].

Let X be a reduced complex space of pure dimension n. We denote $\mathcal{O}_X$ the sheaf of holomorphic functions, Ω_X^q the sheaf of holomorphic q-forms (in particular, $\Omega_X^0 = \mathcal{O}_X$), $\mathcal{M}_X^q$ the sheaf of meromorphic q-forms. First, there is, canonically associated to any meromorphic q-form Ψ on X, a current of bidegree $(q,0)$, denoted $P(\Psi)$ or $[\Psi]$, and called the *principal value* of Ψ, which satisfies the following properties [2]:

1. If Ψ is holomorphic, $[\Psi]$ coincides with the classical current associated to Ψ:
$$[\Psi](\phi) = \int_X \phi \wedge \Psi.$$
2. Let us denote $\tilde{\mathcal{C}}_X^{q,0}$ the sheaf of currents which can be locally written as $[\omega]$, with ω a meromorphic q-form, and $\mathcal{C}_X^{q,0}$ the subsheaf of those which are $\bar\partial$-closed. Then, P induces an isomorphism of $\mathcal{O}_X$-modules, $\mathcal{P}: \mathcal{M}_X^q \to \tilde{\mathcal{C}}_X^{q,0}$, and thus, if we denote by ω_X^q the subsheaf of $\mathcal{M}_X^q$ defined by the meromorphic q-forms which have a $\bar\partial$-closed principal value (called *abelian* forms), an isomorphism $\omega_X^q \simeq \mathcal{C}_X^{q,0}$. Moreover, the natural operator $\partial: \mathcal{M}_X^q \to \mathcal{M}_X^{q+1}$ commutes with $\mathcal{P}$: $[\partial\Psi] = \partial[\Psi]$.

More generally, if Y is a closed analytic subset of pure dimension on X, and ω a meromorphic q-form on Y, we define: $\omega \wedge [Y](\phi) := [\omega](\phi_{|Y})$. Such currents on X are called *principal value* currents.

Now, let $Y_1, \dots, Y_p$ be p hypersurfaces in complete intersection position, that is $Y_1 \cap \dots \cap Y_s$ is of pure codimension s for $1 \leqslant s \leqslant p$. We also assume that the Y_i are locally principal, i.e. locally defined by holomorphic functions. Then, there is on X a *multiple residue* $\mathrm{Res}_{Y_1,\dots,Y_p}[\Psi]$ for a meromorphic q-form Ψ, defined as follows. First, if U is a Stein open subset, and $Y_i = \{f_i = 0\}$ $(1 \leqslant i \leqslant p)$, and $\mathrm{Pol}(\Psi) \setminus Y_1 \cup \dots \cup Y_p \subset \{f_{p+1} = 0\}$, for holomorphic functions $f_1, \dots, f_{p+1}$ on U, we define for a form ϕ with compact support in U:

$$\mathrm{Res}_{Y_1,\dots,Y_p}[\Psi](\phi) = \lim_{m\to\infty} \int_{|f_1|=\epsilon_{m,1},\dots,|f_p|=\epsilon_{m,p},|f_{p+1}|\geqslant\epsilon_{m,p+1}} \Psi \wedge \phi,$$

where the limit is taken on a sequence of $\epsilon_m = (\epsilon_{m,1}, \dots, \epsilon_{m,p+1})$ such that $\epsilon_{m,1} \to 0$, $\epsilon_{m,i+1}/\epsilon_{m,i}^k \to 0$ $(1 \leqslant i \leqslant p)$ for any integer k. A theorem of [2] is that it gives a current, independent of the choice of ϵ_m and of the holomorphic functions $f_1, \dots, f_{p+1}$. Then, we define the current $\mathrm{Res}_{Y_1,\dots,Y_p}[\Psi]$ on the whole X by partition of unity. Such a current is called *residual current*. It is a current of bidegree (q,p), whose support in a reunion of components of $Y_1 \cap \dots \cap Y_p$. Moreover, if $\mathrm{Pol}(\Psi) \subset Y_1 \cup \dots \cup Y_p$, we have:

$$\mathrm{Res}_{Y_1,\dots,Y_p}[\Psi] = \bar\partial\, \mathrm{Res}_{Y_1,\dots,Y_{p-1}}[\Psi].$$

A *locally residual* or *Coleff–Herrera* current, is a current which can be locally written as residual current (we do not assume here (as for instance in [4]) that a locally residual current is $\bar\partial$-closed). We can see that the support of a locally residual current of bidegree (q,p) is an analytic subset of pure codimension p.

By [2] the principal value currents on X $\omega \wedge [Y]$, with Y closed analytic subset of pure dimension and ω meromorphic form on Y, are locally residual. If $Y' \subset Y$ is the polar hypersurface of ω (outside which ω is $\bar\partial$-closed), $\bar\partial(\omega \wedge [Y])$ is a locally residual current of bidegree $(r+p, p+1)$ with support in Y'. We say that ω has *logarithmic pole* on Y' if this current can still be written $\omega' \wedge [Y']$, with ω' a meromorphic $(r-1)$-form on Y'.

Now let Y be an analytic subset of pure codimension p. Then, let us denote $\mathcal{C}_Y^{q,p}$ the sheaf which associates to any open subset U the set $\mathcal{C}_Y^{q,p}(U)$ of $\bar\partial$-closed locally residual currents of bidegree (q, p), with support contained in Y. Let Z be a hypersurface of X intersecting Y properly, i.e. such that the analytic subset $Z \cap Y$ is of pure codimension $p+1$. We denote $\mathcal{C}_Y^{q,p}(*Z)$ the sheaf of locally residual currents of bidegree (q, p) supported in Y, $\bar\partial$-closed outside Z.

Let us assume that Y is locally complete intersection. Then, we will see that $\mathcal{C}_Y^{q,p}$ and $\mathcal{C}_Y^{q,p}(*Z)$ are $\mathcal{O}_X$-modules, and that we have the short exact sequence:

$$0 \to \mathcal{C}_Y^{q,p} \to \mathcal{C}_Y^{q,p}(*Z) \to \mathcal{C}_{Y\cap Z}^{q,p+1} \to 0. \tag{1}$$

We associate to a current T and a smooth form ω the current $\omega \wedge T(\phi) = T(\phi \wedge \omega)$, so that we can write formally: $T(\phi) = \int_X \phi \wedge T$.

We denote ψ_Y^r the subsheaf of $\mathcal{C}_Y^{r+p,p}$ of those currents which can be written on the open subset $U \subset X$ as $\omega \wedge [Y \cap U]$, for ω an abelian r-form on $Y \cap U$. We denote $\psi_Y^r(Z)$ the subsheaf of $\mathcal{C}_Y^{r+p,p}(*Z)$, given by those meromorphic r-forms on Y, abelian outside Z, and having logarithmic pole on Z.

Let us now assume that X is a projective variety, and that $Y_1, \ldots, Y_n$ are analytic hypersurfaces. We assume that the Y_i are *positive*, in the sense that the corresponding Cartier divisors are ample. We also assume that the $Y_1, \ldots, Y_n$ are in complete intersection position. In particular, $Y_1 \cap \cdots \cap Y_n$ is a finite set of points.

Theorem 1.

1. *The complex*:

$$\begin{aligned} 0 \to \omega_X^q \to \mathcal{C}^{q,0}(*Y_1) \xrightarrow{\bar\partial} \mathcal{C}_{Y_1}^{q,1}(*Y_2) \xrightarrow{\bar\partial} \cdots \\ \xrightarrow{\bar\partial} \mathcal{C}_{Y_1\cap\cdots\cap Y_{n-1}}^{q,n-1}(*Y_n) \xrightarrow{\bar\partial} \mathcal{C}_{Y_1\cap\cdots\cap Y_n}^{q,n} \to 0 \end{aligned}$$

   is an acyclic resolution of ω_X^q by $\mathcal{O}_X$-modules, and thus we have a canonical isomorphism for all i, $0 \leqslant i \leqslant n$:

$$H^i(\omega_X^q) \simeq H^0(\mathcal{C}_{Y_1\cap\cdots\cap Y_i}^{q,i}) / \bar\partial H^0(\mathcal{C}_{Y_1\cap\cdots\cap Y_{i-1}}^{q,i-1}(*Y_i)).$$

2. *Moreover, an element $T = H^0(\mathcal{C}_{Y_1\cap\cdots\cap Y_{i-1}}^{q,i-1}(*Y_i))$ can be written as a global residue*: $T = \mathrm{Res}_{Y_i,\ldots,Y_{i-1}}(\Psi)$, *with Ψ a meromorphic q-form with poles contained in $Y_1 \cup \cdots \cup Y_i$; and T is $\bar\partial$-exact iff we can choose Ψ with poles on $Y_1 \cup \cdots \cup Y_{i-1}$.*

Let us assume now moreover that X is smooth and that the $Y_1 \cap \cdots \cap Y_p$ $(1 \leqslant p \leqslant n)$ are reduced complete intersections of pure codimension p. Let us notice that the operators: $\bar\partial : \psi_{Y_1\cap\cdots\cap Y_{p-1}}^{n-p+1}(Y_p) \to \psi_{Y_1\cap\cdots\cap Y_p}^{n-p}$ define a subcomplex of the preceding one for $q = n$.

Then we have the following variant of the preceding theorem:

Theorem 2.

1. *The complex*:

$$0 \to \omega_X^n \to \omega_X^n(Y_1) \xrightarrow{\bar\partial} \psi_{Y_1}^{n-1}(Y_2) \to \cdots$$
$$\xrightarrow{\bar\partial} \psi^1_{Y_1\cap\cdots\cap Y_{n-1}}(Y_n) \xrightarrow{\bar\partial} \psi^0_{Y_1\cap\cdots\cap Y_n} \to 0$$

is an acyclic resolution of ω_X^n *by* $\mathcal{O}_X$*-modules, and thus we have a canonical isomorphism for all* $i, 0 \leqslant i \leqslant n$:

$$H^i(\omega_X^n) \simeq H^0(\psi^{n-i}_{Y_1\cap\cdots\cap Y_i})/\bar\partial H^0(\psi^{n-i}_{Y_1\cap\cdots\cap Y_{i-1}}(Y_i)).$$

2. *Moreover, an element* $T \in H^0(\psi^{n-i}_{Y_1\cap\cdots\cap Y_{i-1}}(Y_i))$ *can be written as a global residue*: $T = \operatorname{Res}_{Y_i,\dots,Y_{i-1}}(\Psi)$, *with* Ψ *a meromorphic closed* n*-form with simple poles on* $Y_1 \cup \cdots \cup Y_i$; *and* T *is* d*-exact iff we can choose* Ψ *with poles on* $Y_1 \cup \cdots \cup Y_{i-1}$.

Remarks. 1. For $q < n$, it is not true in general that the complex with logarithmic poles computes the Dolbeault cohomogy groups, since it is not in general acyclic. The acyclicity for the logarithmic poles, for $q = n$, comes from the Kodaira annihilation theorem.

2. In each of the two preceding theorems, the first part is a variant of Dolbeault's theorem, representing cohomology classes by $\bar\partial$-closed currents with fixed supports. The theorems would remain true if we don't fix the supports in given complete intersections, but consider more general complexes of locally residual currents (resp. of principal value currents of meromorphic forms with logarithmic poles) with any supports. For instance, let us assume X smooth, and denote ψ^{n-i} (resp. $\psi^{\widetilde{n-i}}$) the sheaves of $\bar\partial$-closed principal value currents of bidegree (n,i) (resp. principal value currents of bidegree (n,i) with logarithmic poles). Then, we have by Dolbeault's theorem a natural morphism:

$$H^0(\psi^{n-i})/\bar\partial H^0(\psi^{\widetilde{n-i+1}}) \to H^i(\omega_X^n).$$

This morphism is clearly surjective, since by the preceding theorem we know that we even can fix the supports. But it is also injective: in fact, if the image of $T \in H^0(\psi^{n-i})$ is zero, we know that by definition of the morphism, the current T is $\bar\partial$-exact; and we can include the support of T, which is of pure codimension i, in a reduced complete intersection of i positive hypersurfaces $Y_1,\dots,Y_i$. Thus we can apply again the preceding theorem, to write T in the form $\bar\partial\omega' \wedge [Y']$, with $Y' = Y_1 \cap \cdots \cap Y_{i-1}$, and ω' with logarithmic pole on Y_i. Thus we get, expressed in another way, the main theorem of [8].

3. In the first theorem, we could announce the same theorem, assuming X is compact algebraic and the complements $X \setminus Y_i$ are affine.

We get as corollary a theorem of P. Griffiths [6]:

Corollary 1. *Let* $Y_1,\dots,Y_n$ *be* n *positive hypersurfaces, with* $Y_1 \cap \cdots \cap Y_p$ $(1 \leqslant p \leqslant n)$ *reduced complete intersections, intersecting in* s *distinct points* $P_1,\dots,P_s$, *and let* $c_1,\dots,c_s$ *be* s *complex numbers. A necessary and sufficient condition for the existence of a meromorphic* n*-form* Ψ, *with simple pole on* $Y_1 \cup \cdots \cup Y_n$, *and*

$$(\forall i, 1 \leqslant i \leqslant s) \quad \operatorname{Res}^{P_i}_{Y_1,\dots,Y_n} \Psi = c_i,$$

is that $\sum_{i=1}^s c_i = 0$.

Proof. The existence of Ψ is equivalent of the existence of Ψ such that: $\mathrm{Res}_{Y_1,\dots,Y_n}(\Psi) = \sum_{i=1}^{s} c_i[P_i]$. Thus, the existence of Ψ imply that the "evaluation" current $T = \sum_{i=1}^{s} c_i[P_i]$, which associates to a function f the sum $\sum_{i=1}^{s} f(P_i)$, is exact, and thus annihilates on 1, which means $\sum_{i=1}^{s} c_i = 0$. Reciprocally, if the sum is zero, then the current $T = \sum_{i=1}^{s} c_i[P_i]$ is exact, since X is smooth and connected. Thus by the last theorem, it can be written as a global residue $T = \mathrm{Res}_{Y_1,\dots,Y_n}\Psi$, with Ψ having simple pole on $Y_1 \cup \dots \cup Y_n$. □

2. Secondary results and proofs of the main theorems

2.1. Moderate cohomogical residue operator

Let us now assume that X is a reduced complex analytic space of pure dimension n.

Let Z be a closed analytic subset on X, defined by the sheaf of ideals $\mathcal{I}_Z$, and $Z' \subset Z$ another closed analytic subset. We define the following functor, in the category of $\mathcal{O}_X$-modules:

$$\Gamma_{[Z\setminus Z']}(\mathcal{F}) := \overrightarrow{\lim}_k \mathcal{H}\mathrm{om}_{\mathcal{O}_X}\big(\mathcal{I}^k_{Z'}/\mathcal{I}^k_Z, \mathcal{F}\big).$$

This is a left-exact functor; we define $\mathcal{H}^i_{[Z\setminus Z']}$ as the right derived functor of this functor.

Let us assume Y is a hypersurface, such that for each $x \in Y$, $\mathcal{I}_{Y,x}$ is generated by a germ of holomorphic function f_x. Then, $\Gamma_{[X\setminus Y]}(\mathcal{F})_x$ can be identified by the subset of germs of $\Gamma_{X\setminus Y}\mathcal{F}_x$, which multiplied by a power of f_x, extend through Y to a germ of $\mathcal{F}_x$; we identify two germs if their difference is annihilated by a power of f_x. We denote also $\mathcal{F}(*Y) := \Gamma_{[X\setminus Y]}(\mathcal{F})$. Then we have:

Lemma 1. *$\mathcal{F} \to \mathcal{F}(*Y)$ is an exact functor on $\mathcal{O}_X$-modules.*

Proof. We have that $\mathcal{I}^k_Y$ is coherent, and locally free, since it is generated by one element f^k_x at each point x. Thus, since for a free A-module M, the functor $\mathrm{Hom}_A(M, \bullet)$ is exact, we have that $\mathcal{F} \mapsto \mathcal{H}\mathrm{om}(\mathcal{I}^k_Y, \mathcal{F})$ is exact. By the injective (directed) limit, we get that $\Gamma_{[X\setminus Y]}$ is also exact. □

In particular: $\mathcal{H}^i_{[Z\setminus Y]}(\mathcal{F}) = \mathcal{H}^i_{[Z]}(\mathcal{F})(*Y)$.

Lemma 2. *Let Z be a closed analytic subset, and $Z' \subset Z$ another closed analytic subset. Then we have a natural long exact sequence*:

$$\begin{aligned} 0 &\to \mathcal{H}^0_{[Z']}(\mathcal{F}) \to \mathcal{H}^0_{[Z]}(\mathcal{F}) \to \mathcal{H}^0_{[Z\setminus Z']}(\mathcal{F}) \xrightarrow{\delta_0} \mathcal{H}^1_{[Z']}(\mathcal{F}) \to \cdots \\ &\to \mathcal{H}^i_{[Z']}(\mathcal{F}) \to \mathcal{H}^i_{[Z]}(\mathcal{F}) \to \mathcal{H}^i_{[Z\setminus Z']}(\mathcal{F}) \\ &\xrightarrow{\delta_i} \mathcal{H}^{i+1}_{[Z']}(\mathcal{F}) \to \cdots. \end{aligned}$$

Proof. Let us consider a resolution of $\mathcal{F}$ by injective $\mathcal{O}_X$-modules:

$$0 \to \mathcal{F} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots.$$

Then we have, applying the functor $\mathcal{H}om_{\mathcal{O}_X}(\bullet, \mathcal{I}_i)$ to the short exact sequence $0 \to \mathcal{I}^k_{Z'}/\mathcal{I}^k_Z \to \mathcal{O}_X/\mathcal{I}^k_Z \to \mathcal{O}_X/\mathcal{I}^k_{Z'} \to 0$, an exact sequence of complexes:

$$0 \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}\big(\mathcal{O}_X/\mathcal{I}^k_{Z'}, \mathcal{I}_i\big) \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}\big(\mathcal{O}_X/\mathcal{I}^k_Z, \mathcal{I}_i\big) \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}\big(\mathcal{I}^k_{Z'}/\mathcal{I}^k_Z, \mathcal{I}_i\big) \to 0.$$

For a given i, we have an injective system of sheaves, indexed by the integers k; moreover, this system is directed. Thus, the direct limit on k remains a short exact sequence, and we obtain an exact sequence of complexes:

$$0 \to \Gamma_{[Z']}(\mathcal{I}_i) \to \Gamma_{[Z]}(\mathcal{I}_i) \to \Gamma_{[Z\setminus Z']}(\mathcal{I}_i) \to 0.$$

Since the cohomologies of these complexes compute, by definition, the moderate cohomology sheaves, the long exact sequence associated to this short exact sequence gives us the announced long exact sequence. $\square$

Let Z be an analytic subset of X of pure codimension p, and Y, Y' two hypersurfaces on X locally principal. By taking $Z' := Z \cap Y$ in the preceding long exact sequence, we get as connection operator an operator:

$$\mathcal{H}^i_{[Z\setminus Y]}\big(\omega^q_X\big) \to \mathcal{H}^{i+1}_{[Z\cap Y]}\big(\omega^q_X\big).$$

By applying the exact functor $\Gamma_{[X\setminus Y']}$ we deduce another operator, called *moderate residue operator*:

$$\operatorname{Res}_Y : \mathcal{H}^i_{[Z\setminus (Y\cup Y')]}\big(\omega^q_X\big) \to \mathcal{H}^{i+1}_{[(Z\cap Y)\setminus Y']}\big(\omega^q_X\big).$$

Let us consider the resolution:

$$0 \to \omega^q_X \to \mathcal{D}^{q,0} \xrightarrow{\bar\partial} \mathcal{D}^{q,1} \xrightarrow{\bar\partial} \mathcal{D}^{q,2} \to \cdots$$

by currents.

We know the following (see [7]):

Lemma 3. *The sheaves $\mathcal{D}^{q,p}$ have $\mathcal{O}_{X,x}$-injective fibers.*

We thus deduce the short exact sequences:

$$0 \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}\big(\mathcal{O}_X/\mathcal{I}^k_{Z\cap Y}, \mathcal{D}^{q,p}\big) \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}\big(\mathcal{O}_X/\mathcal{I}^k_Z, \mathcal{D}^{q,p}\big)$$
$$\to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}\big(\mathcal{I}^k_{Z\cap Y}/\mathcal{I}^k_Z, \mathcal{D}^{q,p}\big) \to 0,$$

thus by direct limit:

$$0 \to \Gamma_{[Z\cap Y]}(\mathcal{D}^{q,p}) \to \Gamma_{[Z]}(\mathcal{D}^{q,p}) \to \Gamma_{[Z\setminus Y]}(\mathcal{D}^{q,p}) \to 0;$$

these exact sequences commute with the operator $\bar\partial$, thus we get an exact sequence of complexes:

$$0 \to \Gamma_{[Z\cap Y]}(\mathcal{D}^{q,\bullet}) \to \Gamma_{[Z]}(\mathcal{D}^{q,\bullet}) \to \Gamma_{[Z\setminus Y]}(\mathcal{D}^{q,\bullet}) \to 0.$$

Thus we get a long exact sequence on the cohomologies of these complexes. But we have:

Lemma 4. *These complexes compute the moderate cohomology sheaves*:

$$\mathcal{H}^i_{[Z\cap Y]}\big(\omega^q_X\big),\quad \mathcal{H}^i_{[Z]}\big(\omega^q_X\big),\quad \mathcal{H}^i_{[Z\setminus Y]}\big(\omega^q_X\big).$$

Proof. Let us first show that $\mathcal{H}^i_{[Z]}(\mathcal{D}^{q,p}) = 0$ for $i \geqslant 1$. Since the sheaves $\mathcal{D}^{q,p}$ have injective fibers, the $\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k_Z, \mathcal{D}^{q,p})$ are zero, since $\mathcal{O}_X/\mathcal{I}^k_Z$ being coherent, the fibers commute with the Ext. But since the injective system indexed by the integers k is directed, the cohomology of the direct limit coincide with the direct limit of the cohomology, that is, $\varinjlim_k \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^k, \mathcal{D}^{q,p}) = \mathcal{H}^i_{[Z]}(\mathcal{D}^{q,p})$, which is thus zero. Thus we have also $\mathcal{H}^i_{[Z\cap Y]}(\mathcal{D}^{q,p}) = 0$, and by the preceding long exact sequence also $\mathcal{H}^i_{[Z\setminus Y]}(\mathcal{D}^{q,p}) = 0$.

The sheaves $\mathcal{D}^{q,p}$ are thus $\Gamma_{[Z]}$-acyclic, and thus compute the moderate cohomology sheaves of ω_X^q. $\square$

Thus, in the long exact sequence:

$$0 \to \mathcal{H}^0_{[Z\cap Y]}(\omega_X^q) \to \mathcal{H}^0_{[Z]}(\omega_X^q) \to \mathcal{H}^0_{[Z]}(\omega_X^q)(*Y) \overset{\mathrm{Res}_Y}{\to} \mathcal{H}^1_{[Z\cap Y]}(\omega_X^q) \to \cdots$$

the residues Res_Y can in fact, if we compute the cohomogy groups by currents, be computed as $\bar\partial$-operator.

2.2. Relations with the sheaves of locally residual currents

Let us now assume $Y_1, \dots, Y_p$ hypersurfaces, locally principal, and $Y_1 \cap \cdots \cap Y_k$ of pure codimension k for $1 \leqslant k \leqslant p$.

Let us denote $Y = Y_1 \cap \cdots \cap Y_p$, $Y^i = Y_1 \cup \cdots \cup Y_{i-1} \cup Y_{i+1} \cup \cdots \cup Y_p$.

Remark. Notice that we can obtain the multiple residue operator $\mathrm{Res}_{Y_1,\dots,Y_p}$ as *composed* residue operator: $\mathrm{Res}_{Y_1,\dots,Y_p} = \mathrm{Res}_{Y_1} \circ \cdots \circ \mathrm{Res}_{Y_p}$.

Then we have by [4]:

Lemma 5. *If we denote $\mathcal{Q} := \omega^q(*Y_1 \cup \cdots \cup Y_p)/(\sum_{i=1}^p \omega^q(*Y^i))$, then* $\mathrm{Res}_{Y_1,\dots,Y_p}$ *induces the isomorphisms: $\mathcal{Q} \simeq \mathcal{H}^p_{[Y]}(\omega_X^q) \simeq \mathcal{C}_Y^{q,p}$.*

In particular, any germ of residual current with support in Y $T = H^0(\mathcal{C}^{q,p}_{Y,x})$ can be written as a residue $T = \mathrm{Res}_{Y_1,\dots,Y_p}[\Psi] = \bar\partial\,\mathrm{Res}_{Y_1,\dots,Y_{p-1}}[\Psi]$, with Ψ a germ of meromorphic form with $\mathrm{Pol}(\Psi) \subset Y_1 \cup \cdots \cup Y_p$. Moreover, such a residue is $\bar\partial$ of a current with support in Y iff $\Psi = \sum_{i=1}^p \Psi_i$, with $\mathrm{Pol}(\Psi_i) \subset Y^i$.

Let us now consider Y an analytic subset of pure codimension p, locally complete intersection, and Z a hypersurface intersecting Y properly.

We have by the preceding a natural application $\phi : \mathcal{C}_Y^{q,p} \to \mathcal{H}^p_{[Y]}(\omega_X^q)$, by associating to a $\bar\partial$-closed residual current with support in Y his class; and this map is bijective at each stalk. Thus $\mathcal{C}_Y^{q,p}$ is an $\mathcal{O}_X$-module, and ϕ gives an isomorphism of $\mathcal{O}_X$-modules.

Since the functor $\Gamma_{[X\setminus Z]}$ is exact, we also have an isomorphism of $\mathcal{O}_X$-modules:

$$\phi' : \mathcal{C}_Y^{q,p}(*Z) \simeq \mathcal{H}^p_{[Y\setminus Z]}(\omega_X^q) \simeq \mathcal{H}^p_{[Y]}(\omega_X^q)(*Z).$$

Moreover:

Lemma 6. *$\mathcal{C}_Y^{q,p}(*Z)$ coincides with the sheaf of locally residual currents $\bar\partial$-closed outside Z.*

Proof. We first show that for any germ of locally residual current with support in Y $\alpha \in \tilde{\mathcal{C}}^{q,p}_{Y,x}$ which is $\bar\partial$-closed outside Z, then, if f_x is a defining function for Z at x, there is an integer k such that $f_x^k \alpha$ extends through Z to a germ of $\bar\partial$-closed residual current with support in Y.

For this, it suffices to show by taking $\bar\partial\alpha$ that if β is a germ current with support contained in $\{f_x=0\}$, there is an integer k such that $f_x^k\beta=0$. This is a standard result of current theory by [9, Theorem XXXIV, Chap. 3], after noticing that the closed hypersurface $\{f_x=0\}$ is regular in the sense of [9]. Thus we have, by taking restriction outside Z, a map from germs of residual currents with support in Y, $\bar\partial$-closed outside Z to $\mathcal{C}^{q,p}_{Y,x}(*Z)$. This map is clearly injective, since a residual current of bidegree (q,p) with support of codimension $p+1$ is necessarily 0. It is also surjective, by the formula:

$$f_x^k \operatorname{Res}_{Y_{1,x},\dots,Y_{p,x}} \Psi/f_x^k = \operatorname{Res}_{Y_{1,x},\dots,Y_{p,x}} \Psi,$$

with $Y_x = Y_{1,x}\cap\dots\cap Y_{p,x}$. □

Since Z intersects Y properly, we have $\mathcal{H}^p_{[Z\cap Y]}(\omega_X^q)=0$. Since Y is a locally complete intersection, we have $\mathcal{H}^{p+1}_{[Y]}(\omega_X^q)=0$, and by the preceding long exact sequence, we get the following short exact sequence:

$$0\to\mathcal{H}^p_{[Y]}\big(\omega_X^q\big)\to\mathcal{H}^p_{[Y]}\big(\omega_X^q\big)(*Z)\to\mathcal{H}^{p+1}_{[Z\cap Y]}\big(\omega_X^q\big)\to 0.$$

And thus, we have also the following exact sequence of $\mathcal{O}_X$-modules:

$$0\to\mathcal{C}^{q,p}_Y\to\mathcal{C}^{q,p}_Y(*Z)\to\mathcal{C}^{q,p+1}_{Z\cap Y}\to 0,$$

where the second operator is given by $\bar\partial$. Moreover, if Z' is another hypersurface in X locally principal, intersecting $Z\cap Y$ properly, we deduce a surjective operator:

$$\operatorname{Res}'_Z:\mathcal{C}^{q,p}_Y\big(*(Z\cup Z')\big)\to\mathcal{C}^{q,p+1}_{Z\cap Y}(*Z'),$$

which coincide with the preceding $\bar\partial$ on the subsheaf $\mathcal{C}^{q,p}_Y(*Z)$.

Let us again assume that $Y_1,\dots,Y_p$ are locally principal hypersurfaces, in complete intersection position. By the preceding, the operator $\bar\partial$ defines short exact sequences:

$$0\to\mathcal{C}^{q,i}_{Y_1\cap\dots\cap Y_i}\to\mathcal{C}^{q,i}_{Y_1\cap\dots\cap Y_i}(*Y_{i+1})\to\mathcal{C}^{q,i+1}_{Y_1\cap\dots\cap Y_{i+1}}\to 0,$$

and thus, by iteration, the long exact sequence:

$$\begin{aligned}&0\to\omega_X^q\to\mathcal{C}^{q,0}(*Y_1)\to\mathcal{C}^{q,1}_{Y_1}(*Y_2)\to\cdots\\&\quad\to\mathcal{C}^{q,p-1}_{Y_1\cap\dots\cap Y_{p-1}}(*Y_p)\to\mathcal{C}^{q,p}_{Y_1\cap\dots\cap Y_p}\to 0\end{aligned}\tag{2}$$

with as morphisms the $\bar\partial$-operator on currents.

Lemma 7. *The following complex*:

$$\begin{aligned}&0\ \to\ \omega_X^q\to\bigoplus_{i=1}^p\omega_X^q(*Y_i)\xrightarrow{\delta^1}\cdots\xrightarrow{\delta^{p-2}}\bigoplus_{i=1}^p\omega_X^q(*Y^i)\\&\quad\xrightarrow{\delta^{p-1}}\omega_X^q\big(*(Y_1\cup\dots\cup Y_p)\big)\xrightarrow{\operatorname{Res}_{Y_1,\dots,Y_p}}\mathcal{C}^{q,p}_{Y_1\cap\dots\cap Y_p}\to 0\end{aligned}\tag{3}$$

where the last operator is the multiple residue operator, and the other operators δ^i are the alternate sums, is exact. By allowing poles on a locally principal hypersurface Y_{p+1}, we get another exact sequence:

$$0 \to \omega_X^q(*Y_{p+1}) \to \bigoplus_{i=1}^{p} \omega_X^q\big(*(Y_i \cup Y_{p+1})\big) \to \cdots \to \bigoplus_{i=1}^{p} \omega_X^q\big(*\big(Y^i \cup Y_{p+1}\big)\big)$$
$$\to \omega_X^q\big(*(Y_1 \cup \cdots \cup Y_{p+1})\big) \to \mathcal{C}^{q,p}_{Y_1\cap\cdots\cap Y_p}(*Y_{p+1}) \to 0. \tag{3'}$$

Proof. First, it is clear that this is a complex: the succession of two alternate sums is zero, and the composed residue $\mathrm{Res}_{Y_1,\dots,Y_p}$ is zero on a meromorphic form with pole contained in $Y_1 \cup \cdots \cup Y_{p-1}$. The exactitude on the last step comes from the preceding Lemma 5. At other steps, the exact sequence is shown in [3].

We can modify the preceding exact sequence by applying the exact functor $(*Y_{p+1})$. □

Remark. Let us assume moreover that X is Stein. Thus, any cohomology group of positive degree vanishes automatically for any coherent sheaf. If $\omega_X^q(kZ)$ denotes for any hypersurface Z the sheaf of meromorphic q-forms, abelian outside Z and with poles on Z of order $\leqslant k$, it is coherent and thus acyclic. Let us assume that K is a Stein compact subset of X, and $j > 0$. We deduce, by taking limit for $k \to \infty$: $H^j(K, \omega_X^q(*Z)) = 0$. Since X admits an exhaustion by Stein compact subsets, we deduce that $\omega_X^q(*Z)$ is acyclic on X. Thus, by the preceding exact sequence of Lemma 2, the sheaves $\mathcal{C}^{q,k}_{Y_1\cap\cdots\cap Y_k}(*Y_{k+1})$ are acyclic on X. By the exact sequence (3), we deduce that any $\bar\partial$-closed locally residual current $T = H^0(X, \mathcal{C}_Y^{q,p})$ can be written as a global residue:

$$T = \mathrm{Res}_{Y_1,\dots,Y_p}[\Psi] = \bar\partial\, \mathrm{Res}_{Y_i,\dots,Y_{p-1}}[\Psi],$$

with Ψ a meromorphic q-form with poles on $Y_1 \cup \cdots \cup Y_p$.

Now, let us assume moreover that X is projective, and the hypersurface Y_{p+1} is positive. Then we have the following:

Lemma 8. *For any $s \leqslant p$, $1 \leqslant j_1 < \cdots < j_s \leqslant p$, the sheaves $\omega_X^q(*Y_{j_1} \cup \cdots \cup Y_{j_s} \cup Y_{p+1})$ are acyclic. From the exact sequence (3′), the sheaf $\mathcal{C}^{q,p}_{Y_1\cap\cdots\cap Y_p}(*Y_{p+1})$ is also acyclic.*

Proof. Since the hypersurface Y_{p+1} is positive, we have by Serre's theorem, if we denote L_{p+1} the corresponding Cartier divisor on X, that: $H^j(X, \mathcal{F}(L^k_{p+1})) = 0$ for $k \gg 0$ and $j > 0$, if $\mathcal{F}$ is a coherent sheaf on X. Since X is compact, we can do the passage to the limit when $k \to \infty$ [5], and we get: $H^j(X, \mathcal{F}(*Y_{p+1})) = 0$. Since the sheaves $\omega_X^q(*Y_{j_1} \cup \cdots \cup Y_{j_s})$ are not coherent, we cannot use directly the positivity assumption on Y_{p+1}. But let us first take a limitation of the order of the poles: we get subsheaves $\omega_X^q(tY_{j_1} + \cdots + tY_{j_s})$, which have poles of order $\leqslant t$ along the Y_{j_i}. These subsheaves are coherent, thus we can apply the positivity assumption; we get thus for $j > 0$: $H^j(X, \omega_X^q(tY_{j_1} + \cdots + tY_{j_s})(*Y_{p+1})) = 0$; by taking the limit for t going to infinity, we get: $H^j(X, \omega_X^q(*Y_{j_1} \cup \cdots \cup Y_{j_s} \cup Y_{p+1})) = 0$. Thus we have the acyclicity of all the terms in the exact sequence (3′), except the last one. But then the acyclicity of $\mathcal{C}^{q,p}_{Y_1\cap\cdots\cap Y_p}(*Y_{p+1})$ follows from the fact that a sheaf with an "acyclic" resolution is itself acyclic. □

Proof of the first theorem. Now we assume $Y_1, \dots, Y_n$ are n positive locally principal hypersurfaces in complete intersection position on the projective variety X of dimension n. Since by Lemma 8 the given resolution of ω_X^q is acyclic, it computes its cohomology groups:

$$H^i\big(\omega_X^q\big) \simeq H^0\big(\mathcal{C}^{q,i}_{Y_1\cap\cdots\cap Y_i}\big)/\bar\partial H^0\big(\mathcal{C}^{q,i-1}_{Y_1\cap\cdots\cap Y_{i-1}}(*Y_i)\big).$$

We thus get the first part of Theorem 1.

Moreover, let us consider the exact sequence:

$$0 \to \omega_X^q(*Y_{p+1}) \to \bigoplus_{i=1}^{p} \omega_X^q\big(*(Y_i \cup Y_{p+1})\big) \to \cdots \to \bigoplus_{i=1}^{p} \omega_X^q\big(*\big(Y^i \cup Y_{p+1}\big)\big)$$
$$\to \omega_X^q(*Y_1 \cup \cdots \cup Y_{p+1}) \to \mathcal{C}^{q,p}_{Y_1 \cap \cdots \cap Y_p}(*Y_{p+1}) \to 0.$$

Since the functor of global section Γ is exact on acyclic sheaves, we get a surjective map:

$$\mathrm{Res}_{Y_1,\ldots,Y_p} : H^0\big(\omega_X^q(*Y_1 \cup \cdots \cup Y_{p+1})\big) \to H^0\big(\mathcal{C}^{q,p}_{Y_1 \cap \cdots \cap Y_p}(*Y_{p+1})\big).$$

Then, if $T \in H^0(\mathcal{C}^{q,p}_{Y_1 \cap \cdots \cap Y_p}(*Y_{p+1}))$ can be written as $\mathrm{Res}_{Y_1,\ldots,Y_p}(\Psi)$ with $\mathrm{Pol}(\Psi) \subset Y_1 \cup \cdots \cup Y_p$, it is also: $T = \bar\partial(\mathrm{Res}_{Y_1,\ldots,Y_{p-1}}(\Psi))$, and thus T is $\bar\partial$-exact. Reciprocally, if T is $\bar\partial$-exact, then its class in $H^p(\omega_X^q)$ is zero, and thus by the first part of the theorem, it is $T = \bar\partial T'$, with $T' \in H^0(\mathcal{C}^{q,p-1}_{Y_1 \cap \cdots \cap Y_{p-1}}(*Y_p))$; but then, $T' = \mathrm{Res}_{Y_1,\ldots,Y_{p-1}}(\Psi)$, with Ψ a meromorphic q-form with poles contained in $Y_1 \cup \cdots \cup Y_p$; and thus, $T = \mathrm{Res}_{Y_1,\ldots,Y_p}(\Psi)$. This achieves the proof of the Theorem 1. □

Proof of the second theorem. We now assume moreover that the projective variety X is smooth, and that the hypersurfaces $Y_1, \ldots, Y_n$ define *reduced* complete intersections. □

We can show in the same way as above the following:

Lemma 9. *The following complexes with simple poles is exact*:

$$0 \to \Omega_X^n \to \bigoplus_{i=1}^{p} \Omega_X^n(Y_i) \to \cdots \to \bigoplus_{i=1}^{p} \Omega_X^n(Y^i)$$
$$\to \Omega_X^n(Y_1 \cup \cdots \cup Y_p) \xrightarrow{\mathrm{Res}_{Y_1,\ldots,Y_p}} \psi^{n-p}_{Y_1 \cap \cdots \cap Y_p} \to 0 \tag{4}$$

and, by tensoring by the line bundle associated with Y_{p+1}:

$$0 \to \Omega_X^n(Y_{p+1}) \to \bigoplus_{i=1}^{p} \Omega_X^n(Y_i \cup Y_{p+1}) \to \cdots \to \bigoplus_{i=1}^{p} \Omega_X^n\big(Y^i \cup Y_{p+1}\big)$$
$$\to \Omega_X^n(Y_1 \cup \cdots \cup Y_{p+1}) \xrightarrow{\mathrm{Res}_{Y_1,\ldots,Y_p}} \psi^{n-p}_{Y_1 \cap \cdots \cap Y_p}(Y_{p+1}) \to 0. \tag{4'}$$

But we know, by the vanishing theorem of Kodaira, that the sheaves $\Omega_X^n(Y)$, for Y a hypersurface containing a positive hypersurface, is acyclic, and thus also the sheaves $\psi^{n-p}_{Y_1 \cap \cdots \cap Y_p}(Y_{p+1})$. By taking global sections, we see that any $T \in H^0(\psi^{n-p}_{Y_1 \cap \cdots \cap Y_p}(Y_{p+1}))$ can be written as a global residue of a meromorphic form with simple pole on $Y_1 \cup \cdots \cup Y_{p+1}$.

Finally, the exact sequence (4): shows that if the current $T \in H^0(\psi^{n-p}_{Y_1 \cap \cdots \cap Y_p})$ gives a zero class in $H^p(\Omega_X^n)$, it can be written as a global residue of a meromorphic form with simple pole on $Y_1 \cup \cdots \cup Y_p$.

The following lemma achieves the proof of the second theorem:

Lemma 10. *The following complex is also an acyclic resolution of Ω_X^n:*

$$\begin{aligned} 0 &\to \Omega_X^n \to \Omega_X^n(Y_1) \to \psi_{Y_1}^{n-1}(Y_2) \to \cdots \\ &\to \psi^1_{Y_1\cap\cdots\cap Y_{n-1}}(Y_n) \to \psi^0_{Y_1\cap\cdots\cap Y_n} \to 0. \end{aligned} \tag{5}$$

Proof. For the exactitude, it suffices to show that, if U is a Stein open subset and $Z_1, \ldots, Z_p$ are analytic hypersurfaces in complete intersection position in U, a $\bar\partial$-closed current $\omega \wedge [Z]$, with $Z = Z_1 \cap \cdots \cap Z_p$ and ω an abelian r-form of maximal degree, can be written $\bar\partial\omega' \wedge [Z']$, with $Z' = Z_1 \cap \cdots \cap Z_{p-1}$ and ω' a meromorphic $(r+1)$-form on Z'. But the following exact sequence:

$$\begin{aligned} 0 &\to \Omega_X^n \to \bigoplus_{i=1}^p \Omega_X^n(Z_i) \to \cdots \to \bigoplus_{i=1}^p \Omega_X^n(Z^i) \\ &\to \Omega_X^n(Z_1 \cup \cdots \cup Z_p) \xrightarrow{\mathrm{Res}_{Z_1,\ldots,Z_p}} \psi^{n-p}_{Z_1\cap\cdots\cap Z_p} \to 0 \end{aligned}$$

is formed with coherent and thus acyclic sheaves on U; thus $\omega \wedge [Z] = \mathrm{Res}_{Z_1,\ldots,Z_p}[\Psi] = \bar\partial\,\mathrm{Res}_{Z_1,\ldots,Z_{p-1}}[\Psi]$, with Ψ simple pole on $Z_1 \cup \cdots \cup Z_p$; and thus $\mathrm{Res}_{Z_1,\ldots,Z_p}[\Psi] = \bar\partial\omega' \wedge [Z']$. Moreover, we know by the preceding that the resolution (5) is acyclic. □

3. Hodge conjecture

Let X be a projective manifold. The Hodge conjecture says that any closed form of bidegree (p,p) with *rational cohomology class* (i.e. such that the integral on any real submanifold of dimension $2p$ is rational) is cohomologous to an integration current $\sum_i c_i[Y_i]$ with rational coefficients.

The Hodge conjecture is equivalent to the following:

Conjecture 1. *Any closed form of bidegree (p,p), with rational class, is cohomologous to a d-closed locally residual current of bidegree (p,p).*

The equivalence is shown as follows. First, Hodge conjecture implies Conjecture 1, since the current $\sum_i c_i[Y_i]$ is locally residual.

Reciprocally, let us assume that Conjecture 1 is true. Let Ψ be a closed form of bidegree (p,p) with rational class. Then, ψ is cohomologous to a d-closed locally residual current T of bidegree (p,p). Let us consider $Y_1, \ldots, Y_p$ hypersurfaces in complete intersection position, with $Z := \mathrm{Supp}(T) \subset Y := Y_1 \cap \cdots \cap Y_p$, and write locally T, in a neighborhood of a point $x \in Z$, as $\mathrm{Res}_{Y_1,\ldots,Y_p}\,\omega$ for some meromorphic ∂-closed p-form ω with $\mathrm{Pol}(\omega) \subset Y_1 \cup \cdots \cup Y_p$. Then we associate to T at x the number $\int_{|f_1|=\epsilon_1,\ldots,|f_p|=\epsilon_p} \omega$, if f_j is a defining functions for Y_j at x. These numbers do not depend of the choices of ω or of the f_i, and are thus a constant c_i along an irreducible component Z_i of Z. We thus associate to T a current $T' = \sum_i c_i[Z_i]$ where Z_i are irreducible components of Z and c_i the numbers just defined. This current T' is cohomologous to T, since it gives the same integral on any real oriented $2p$-dimensional submanifold.

Thus T is cohomologous to a current $T' = \sum_i c_i[Z_i]$, where we can assume that the classes of the currents $[Z_i]$ in $H^{2p}(X,\mathbb{C})$ are $\mathbb{C}$-independent, and thus their classes in $H^{2p}(X,\mathbb{Q})$ are $\mathbb{Q}$-independent. Then, by Poincaré duality, we can find rational cohomology classes $\phi_i \in$

$H^{2n-2p}(X,\mathbb{Q})$ $(1 \leqslant i \leqslant k)$ such that $[Z_i](\phi_j) = \delta_{ij}$. Thus, we get: $T'(\phi_j) = c_j$, and thus the coefficients c_j are rational since T' has rational class. This achieves the proof of equivalence.

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